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SUBSETS OF PRODUCTS OF POSITIVE DENSITY ON VAN DER WAERDEN SETS

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Abstract. We prove that for every sequence $(m_q)_q$ of positive integers and for every real $0<\delta\leqslant 1$ there is a sequence $(n_q)_q$ of positive integers such that for every $D\subseteq\bigcup_k\prod_{q=0}^{k-1}[n_q]$ satisfying

$$\frac{\big|D\cap\prod_{q=0}^{k-1}[n_q]\big|}{\prod_{q=0}^{k-1}n_q}\geqslant\delta$$

for every k in a van der Waerden set, there is a sequence $(J_q)_q$, where J_q is an arithmetic progression of length m_q contained in $[n_q]$ for all q, such that $\prod_{q=0}^{k-1}J_q\subseteq D$ for every k in a van der Waerden set. Moreover, working in an abstract setting, we obtain J_q to be any configuration of natural numbers that can be found in an arbitrary set of positive density.

1. Introduction

In [TT] a density version of a Ramsey theoretic result [DLT, To] has been established. In particular it was shown that for every positive real ε and every sequence $(m_q)_q$ of positive integers there exists a sequence $(n_q)_q$ of positive integers with the following property. For every infinite subset L of the positive integers and every sequence $(D_\ell)_{\ell\in L}$ such that D_ℓ is a subset of $\prod_{q=0}^{\ell-1}\{1,...,n_q\}$ of density at least ε for all $\ell\in L$, there exist a sequence $(I_q)_q$ and infinite subset L' of L such that I_q is a subset of $\{1,...,n_q\}$ of cardinality m_q for all non-negative integers q and the set $\prod_{q=0}^{\ell-1}I_q$ is a subset of D_ℓ for all $\ell\in L'$.

In the present paper we provide a strengthening of this result which is optimal in several aspects. Firstly, we show that the set L' can be chosen to be a van der Waerden set provided that L itself is a van der Waerden set (this is, clearly, a necessary condition). Moreover, the sets I_q can be endowed with additional structure. For instance, each I_q can be chosen to be an arithmetic or a polynomial progression. The construction of the sequence $(I_q)_q$ is effective avoiding, in particular, compactness arguments as in [TT].

In order to state our result we need to introduce some pieces of notation. By $\mathbb{N}$ we denote the set of the non-negative integers; $\mathbb{N}_+$ stands for the set of all positive integers. For every set X by $X^{<\mathbb{N}}$ we denote the set of all finite sequences in X. The empty sequence is denoted by $\emptyset$ and is included in $X^{<\mathbb{N}}$. For every positive

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integer k by $[k]$ we denote the set $\{1,...,k\}$. By convention, $[0]$ stands for the empty set.

Let Y be a (possibly infinite) set. If X is a nonempty finite subset of Y and A is an arbitrary subset of Y, then the *density of A relative to X*, denoted by $\mathrm{dens}_X(A)$, is the quantity defined by

$$\mathrm{dens}_X(A) = \frac{|A \cap X|}{|X|}. \tag{1}$$

Recall that a subset of $\mathbb{N}_+$ is called a *var der Waerden set* if it contains arbitrarily long arithmetic progressions. We will also need the following slight variant of the standard notion of a density regular family (see, e.g., [Mc1, Mc2]).

Definition 1. *A family $\mathcal{F}$ of nonempty finite subsets of $\mathbb{N}_+$ is called* uniformly density regular *if for every $0 < \varepsilon \leqslant 1$ there exists an integer n_0 such that for every interval I of $\mathbb{N}_+$ of length at least n_0 and every subset A of I with $\mathrm{dens}_I(A) \geqslant \varepsilon$, the set A contains an element of $\mathcal{F}$. The least such n_0 will be denoted by $B(\mathcal{F},\varepsilon)$. Finally, the set of all uniformly density regular families will be denoted by $\mathcal{R}$.*

There are several examples of uniformly regular families. The simplest one is the set of all subsets of the positive integers with exactly k elements where k is a fixed positive integer. By the famous Szemerédi Theorem [Sz], the set AP_k of all arithmetic progressions of length k is also a uniformly regular family; see [G] for the best known upper bounds for the numbers $B(\mathrm{AP}_k,\varepsilon)$. Moreover, for every choice of polynomials $p_1,...,p_k$ taking integer values on the integers and zero on zero, the family $\big\{\{a+p_1(n),...,a+p_k(n)\} : a \in \mathbb{Z} \text{ and } n \in \mathbb{Z}\big\}$ is uniformly density regular. This is a consequence of the work of V. Bergelson and A. Leibman [BL]. More examples of uniform density regular families can be found in [FW, BM].

We are now ready to state our main result.

Theorem 2. *Let $0 < \delta \leqslant 1$. Then there exists a map $V_\delta : \mathcal{R}^{<\mathbb{N}} \times \mathbb{N}_+^{<\mathbb{N}} \to \mathbb{N}$ with the following property. For every sequence $(n_q)_q$ of positive integers, every sequence $(\mathcal{F}_q)_q$ of uniformly regular families, every var der Waerden set L and every sequence $(D_k)_{k\in L}$ such that*

(a) $n_0 \geqslant V_\delta\big(\mathcal{F}_0,\emptyset\big)$,
(b) $n_q \geqslant V_\delta\big((\mathcal{F}_p)_{p=0}^{q},(n_p)_{p=0}^{q-1}\big)$ *for all positive integers q and*
(c) D_ℓ *is a subset of* $\prod_{q=0}^{\ell-1}[n_q]$ *of density at least δ for all $\ell \in L$*

there exist a sequence $(I_q)_q$ and a van der Waerden subset L' of L such that

(i) I_q *is an element of $\mathcal{F}_q$ contained in $[n_q]$ for all $q \in \mathbb{N}$ and*
(ii) $\prod_{q=0}^{\ell-1} I_q \subseteq D_\ell$ *for all $\ell \in L'$.*

The definition of the map V_δ is based on an auxiliary map T that we will define in Section 3. The proof of Theorem 2 as well as the definition of the map V_δ are given in Section 4.

2. Correlation of measurable events on arithmetic progressions.

Recall the following two classical combinatorial results due to B. L. van der Waerden [vW] and E. Szemerédi [Sz] respectively.

Theorem 3. *For every r, k positive integers, there exists a positive integer n_0 such that for every $n \geqslant n_0$ and every coloring of $[n]$ into r many colors, one of the colors contains an arithmetic progression of length k. The least such n_0 will be denoted by* $\mathrm{W}(k, r)$.

Theorem 4. *For every $0 < \varepsilon \leqslant 1$ and every positive integer k there exists a positive integer n_0 such that for every $n \geqslant n_0$ and every A subset of $[n]$ of density at least ε, the set A contains an arithmetic progression of length k. The least such n_0 will be denoted by* $\mathrm{S}(k, \varepsilon)$.

Theorem 3 has the following easy consequence, essentially stating that the collection of all van der Waerden sets forms a coideal on $\mathbb{N}_+$.

Fact 5. *One of the colors of a finite coloring of a van der Waerden set is a van der Waerden set.*

We proceed to define some numerical invariants. For every $0 < \eta \leqslant 1$ and every integer $k \geqslant 2$ we set

$$\theta_1(k, \eta) = (\eta/2) \cdot \binom{\mathrm{S}(k, \eta/2)}{2}^{-1}, \tag{2}$$

while for every $0 < \eta \leqslant 1$ we set

$$\theta_1(1, \eta) = \eta. \tag{3}$$

We will need the following lemma.

Lemma 6. *Let $0 < \eta \leqslant 1$ and k be a positive integer. Then for every integer n with $n \geqslant \mathrm{S}(k, \eta/2)$ and every family $(A_i)_{i=1}^n$ of measurable events in a probability space (Ω, Σ, μ) such that $\mu(A_i) \geqslant \eta$ for all $i \in [n]$ there exists an arithmetic progression P of length k contained in $[n]$ such that*

$$\mu\Big(\bigcap_{i \in P} A_i\Big) \geqslant \theta_1(k, \eta). \tag{4}$$

Proof. For $k = 1$ the result is immediate. Thus, let as assume that $k \geqslant 2$. Fix $n \geqslant \mathrm{S}(k, \eta/2)$ and a family $(A_i)_{i=1}^n$ of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_i) \geqslant \eta$ for all $i \in [n]$. We set $n_0 = \mathrm{S}(k, \eta/2)$ and

$$A = \{(i, x) \in [n_0] \times \Omega : x \in A_i\}. \tag{5}$$

Clearly, the product probability measure $\mathrm{dens}_{[n_0]} \otimes \mu$ of A is at least η. For every $x \in \Omega$ let

$$A^x = \{i \in [n_0] : (i, x) \in A\}. \tag{6}$$

By Fubini Theorem, setting

$$C = \Big\{x \in \Omega : \mathrm{dens}_{[n_0]}(A^x) \geqslant \frac{\eta}{2}\Big\}, \tag{7}$$

we have that $C \in \Sigma$ and $\mu(C) \geqslant \frac{\eta}{2}$. By Theorem 4 and the choice of n_0, for every $x \in C$ there exists an arithmetic progression P_x of length k contained in A^x. Observe that for every $x \in C$ we have

$$x \in \bigcap_{i \in P_x} A_i. \tag{8}$$

There are at most $\binom{n_0}{2}$ many arithmetic progressions of length k contained in $[n_0]$. Therefore, there exist an arithmetic progression P of length k contained in $[n_0]$ and a measurable subset C' of C with

$$\mu(C') \geqslant \mu(C) \cdot \binom{\mathrm{S}(k, \eta/2)}{2}^{-1} \overset{(2)}{\geqslant} \theta_1(k, \eta) \tag{9}$$

and such that $P_x = P$ for all $x \in C'$. Invoking (8) we see that $C' \subseteq \cap_{i \in P} A_i$. Hence $\mu(\cap_{i \in P} A_i) \geqslant \mu(C') \overset{(9)}{\geqslant} \theta_1(k, \eta)$ and the proof is completed. □

We will also need a variant of Lemma 6 which is stated in the more general context of uniformly density regular families. To state it we need, first, to introduce some further invariants. Specifically, for every $0 < \eta \leqslant 1$ and every uniformly density regular family $\mathcal{F}$ we set

$$M(\mathcal{F}, \eta) = \max\Big\{|\{F \in \mathcal{F} : F \subseteq I\}| : I \text{ is an interval of length } B(\mathcal{F}, \eta)\Big\} \tag{10}$$

and we define

$$\theta_2(\mathcal{F}, \eta) = \frac{\eta}{4 \cdot M(\mathcal{F}, \eta/4)}. \tag{11}$$

Lemma 7. *Let $0 < \eta \leqslant 1$ and $\mathcal{F}$ be a uniformly density regular family. Also let n be an integer with $n \geqslant (2/\varepsilon) \cdot B(\mathcal{F}, \varepsilon/4)$ and (Ω, Σ, μ) be a probability space. Finally let A be a subset of $[n] \times \Omega$ with $(\mathrm{dens}_{[n]} \otimes \mu)(A) \geqslant \eta$. Then there exists an element F of $\mathcal{F}$ such that, setting $\widetilde{A} = \{x \in \Omega : (i, x) \in A$ for all $i \in F\}$, we have*

$$\mu(\widetilde{A}) \geqslant \theta_2(\mathcal{F}, \eta). \tag{12}$$

Proof. We set $n_0 = B(\mathcal{F}, \eta/4)$. First we pick a subinterval I of $[n]$ of length n_0 such that

$$(\mathrm{dens}_I \otimes \mu)\big(A \cap (I \times \Omega)\big) \geqslant \eta/2 \tag{13}$$

as follows. We set $\ell = \lfloor n/n_0 \rfloor$ and we pick $I_1, ..., I_\ell$ disjoint subintervals of $[n]$ each of length n_0. We set $J = \cup_{j=1}^{\ell} I_j$. By the assumptions on n, we have that $\mathrm{dens}_{[n]}([n] \setminus J) < \eta/2$. Consequently, since $I_1, ..., I_\ell$ are of the same length, we have that

$$\frac{\eta}{2} \leqslant (\mathrm{dens}_J \otimes \mu)\big(A \cap (J \times \Omega)\big) = \frac{1}{\ell} \sum_{j=1}^{\ell} (\mathrm{dens}_{I_j} \otimes \mu)\big(A \cap (I_j \times \Omega)\big). \tag{14}$$

Hence for some $j_0 \in [\ell]$ we have that $(\mathrm{dens}_{I_{j_0}} \otimes \mu)\big(A \cap (I_{j_0} \times \Omega)\big) \geqslant \eta/2$. Let $I = I_{j_0}$.

For every $i \in I$ we set

$$A_i = \{x \in \Omega : (i,x) \in A\} \tag{15}$$

and for every $x \in \Omega$ we set

$$A^x = \{i \in I : (i,x) \in A\}. \tag{16}$$

By (13) and Fubini Theorem, we have that the set

$$C = \Big\{x \in \Omega : \mathrm{dens}_I(A^x) \geqslant \eta/4\Big\} \tag{17}$$

is a measurable event of probability at least $\eta/4$. Since I is of length $B(\mathcal{F}, \eta/4)$, for every $x \in C$ we have that there exists an element F_x of $\mathcal{F}$ contained in A^x. Let us observe that for every $x \in C$, by the definition of the set A^x, we have that

$$x \in \bigcap_{i \in F_x} A_i. \tag{18}$$

Since I contains at most $M(\mathcal{F}, \eta/4)$ elements of $\mathcal{F}$, we have that there exist an element F of $\mathcal{F}$ contained in I and a measurable subset C' of C such that

$$\mu(C') \geqslant \frac{\mu(C)}{M(\mathcal{F}, \eta/4)} \overset{(11)}{\geqslant} \theta_2(\mathcal{F}, \eta) \tag{19}$$

and $F_x = F$ for all $x \in C'$. Invoking (18) we have that $C' \subset \cap_{i \in F} A_i$. Setting $\widetilde{A}$ as in the statement, we clearly have that the intersection $\cap_{i \in F} A_i$ is subset of $\widetilde{A}$. Thus $\mu(\widetilde{A}) \geqslant \mu(\cap_{i \in F} A_i) \geqslant \mu(C') \overset{(18)}{\geqslant} \theta_2(\mathcal{F}, \eta)$ as desired. □

Before we proceed let us introduce some additional notation. Let (Ω, Σ, μ) be a probability space and B be a measurable event of positive probability. For every $A \in \Sigma$ we set

$$\mu_B(A) = \frac{\mu(A \cap B)}{\mu(B)}. \tag{20}$$

We have the following elementary fact.

Fact 8. *Let $0 < \eta, \theta \leqslant 1$. Also let (Ω, Σ, μ) be a probability space and A, B be two measurable events such that $\mu(A) \geqslant \eta$ and $\mu(B) \geqslant \theta$. If $\mu_B(A) \leqslant \eta/2$ then $\mu(\Omega \setminus B) \geqslant \eta/2$ and $\mu_{\Omega\setminus B}(A) \geqslant \eta + \eta\theta/2$.*

Proof. Assuming that $\mu_B(A) \leqslant \eta/2$ we have the following. First we observe that

$$\mu(\Omega \setminus B) \geqslant \mu(A \setminus B) = \mu(A) - \mu(A \cap B) \geqslant \mu(A) - \mu_B(A) \geqslant \eta/2. \tag{21}$$

Since

$$\begin{aligned} \eta \leqslant \mu(A) &= \mu(\Omega \setminus B) \cdot \mu_{\Omega\setminus B}(A) + \mu(B) \cdot \mu_B(A) \\ &\leqslant \big(1 - \mu(B)\big) \cdot \mu_{\Omega\setminus B}(A) + (\eta/2) \cdot \mu(B), \end{aligned} \tag{22}$$

we have that

$$\begin{aligned}\mu_{\Omega\setminus B}(A) &\geqslant \eta\cdot\frac{1-\mu(B)/2}{1-\mu(B)} = \eta\cdot\Big(1+\frac{\mu(B)/2}{1-\mu(B)}\Big)\\ &\geqslant \eta+\eta\cdot\mu(B)/2 \geqslant \eta+\eta\theta/2\end{aligned} \tag{23}$$

as desired. □

Lemma 9. *Let $0<\eta\leq 1$ and k be a positive integer. Also let L be a van der Waerden set and $(A_\ell)_{\ell\in L}$ be a family of measurable events in a probability space (Ω,Σ,μ) such that $\mu(A_\ell)\geqslant\eta$ for all $\ell\in L$. Then there exist an arithmetic progression P of length k contained in L, a van der Waerden subset L' of L and a subset B of $\cap_{i\in P}A_i$ such that*

$$\mu(B)\geqslant\Big(\frac{\eta}{2}\Big)^{\lfloor\frac{2}{\eta\cdot\theta_1(k,\eta)}\rfloor-1}\theta_1(k,\eta) \tag{24}$$

and $\mu_B(A_\ell)\geqslant\eta/2$, for all $\ell\in L'$. Moreover, the set B belongs to the algebra generated by the family $\{A_\ell:\ell\in L \text{ and } \ell\leqslant\max R\}$.

Proof. We set $L_0=L$ and $\Omega_0=\Omega$. We pick a positive integer s_0 with $s_0\leqslant\lfloor\frac{2}{\eta\cdot\theta_1(k,\eta)}\rfloor$ and we construct by induction $L_1,...,L_{s_0}$ and $P_1,...,P_{s_0}$ such that setting inductively $B_t=(\cap_{i\in P_t}A_i)\cap\Omega_{t-1}$ and $\Omega_t=\Omega_{t-1}\setminus B_t$ for all $t=1,...,s_0$, we have that the following are satisfied.

(i) For every $t=1,...,s_0$ we have that L_t is a van der Waerden subset of L_{t-1}.
(ii) For every $t=1,...,s_0$ we have that P_t is an arithmetic progression of length k contained in L_{t-1}.
(iii) For every $t=0,...,s_0-1$ we have $\mu(\Omega_t)\geqslant(\eta/2)^t$.
(iv) For every $t=1,...,s_0$ we have $\mu_{\Omega_{t-1}}(B_t)\geqslant\theta_1(k,\eta)$.
(v) For every $t=0,...,s_0-1$ we have $\mu_{\Omega_t}(A_\ell)\geqslant\eta+t\cdot\eta\cdot\theta_1(k,\eta)/2$ for all $\ell\in L_t$.
(vi) For every $t=1,...,s_0-1$ we have that $\mu_{B_t}(A_\ell)<\eta/2$ for all $\ell\in L_t$.
(vii) $\mu_{B_{s_0}}(A_\ell)\geqslant\eta/2$ for all $\ell\in L_{s_0}$.

Assume that for some $s<\lfloor\frac{2}{\eta\cdot\theta_1(k,\eta)}\rfloor$ we have constructed $(L_t)_{t=0}^s$ and if $s\geqslant 1$, $(P_t)_{t=1}^s$, satisfying (i)-(vi) above. Let $(\Omega_t)_{t=0}^s$ be as defined above. By the inductive assumption (i), we have that L_s is a van der Waerden set and therefore we may pick an arithmetic progression P of length $\mathrm{S}(k,\eta/2)$ contained in L_s. By the inductive assumption (v) and Lemma 6 there exists an arithmetic progression P_{s+1} of length k contained in P such that

$$\mu_{\Omega_s}(B_{s+1})\geqslant\theta_1(k,\eta), \tag{25}$$

where $B_{s+1}=(\cap_{i\in P_{s+1}}A_i)\cap\Omega_s$. By Fact 5 we pass to a van der Waerden subset L_{s+1} of L_s such that either

(a) $\mu_{B_{s+1}}(A_\ell)\geqslant\eta/2$, for all $\ell\in L_{s+1}$, or
(b) $\mu_{B_{s+1}}(A_\ell)<\eta/2$, for all $\ell\in L_{s+1}$.

If (a) occurs, then we set $s_0 = s+1$ and the inductive construction is complete. Let us assume that (b) holds. Then invoking (25) and (v) of the inductive assumptions, by Fact 8, we have that

$$\mu_{\Omega_{s+1}}(A_\ell) \geqslant \eta + (s+1)\cdot \eta \cdot \theta_1(k,\eta)/2 \tag{26}$$

for all $\ell \in L_{s+1}$, where $\Omega_{s+1} = \Omega_s \setminus B_{s+1}$. Moreover, by Fact 8 and the inductive assumptions (iii) and (v) we have that $\mu(\Omega_{s+1}) \geqslant (\eta/2)\cdot\mu(\Omega_s) \geqslant (\eta/2)^{s+1}$. The inductive step of the construction is complete. Finally, let as point out that if $s = \lfloor \frac{2}{\eta\cdot\theta_1(k,\eta)} \rfloor - 1$, then (a) has to occur. Indeed, assuming that (b) occurs then by (26) we would have that the relative probability of A_ℓ inside Ω_{s+1} exceeds 1.

Hence, setting $L' = L_{s_0}$, $P = P_{s_0}$ and $B = B_{s_0}$, we have that L' is a van der Waerden subset of L and P is an arithmetic progression of length k contained in L. Moreover, we have that $B = (\cap_{i\in P} A_i) \cap \Omega_{s_0-1} \subseteq \cap_{i\in P} A_i$ and

$$\begin{aligned}\mu(B) =& \mu(B_{s_0}) = \mu_{\Omega_{s_0-1}}(B_{s_0})\cdot \mu(\Omega_{s_0-1}) \overset{\text{(iv),(iii)}}{\geqslant} \theta_1(k,\eta)\cdot(\eta/2)^{s_0-1}\\ \geqslant& \theta_1(k,\eta)\cdot(\eta/2)^{\lfloor \frac{2}{\eta\cdot\theta_1(k,\eta)}\rfloor - 1}.\end{aligned} \tag{27}$$

By (vii), we have that $\mu_B(A_\ell) \geqslant \eta/2$, for all $\ell \in L'$. Finally, it is immediate that the set B, by its definition, belongs to the algebra generated by the family $\{A_\ell : \ell \in L \text{ and } \ell \leqslant \max R\}$ as desired. □

3. The auxiliary map T

As we have already mentioned the definition of the map V_δ makes use of an auxiliary map T. Recall that by $\mathcal{R}$ we denote the set of all uniformly regular families (see Definition 1). We define the map $T : \mathcal{R}^{<\mathbb{N}} \times \mathbb{R}_+ \to \mathbb{N}$, where by $\mathbb{R}_+$ we denote the set of all positive reals, as follows. Let q be a non-negative integer and $((\mathcal{F}_p)_{p=0}^q, \varepsilon)$ be an element of $\mathcal{R}^{<\mathbb{N}} \times \mathbb{R}_+$. We inductively define $(\varepsilon_p)_{p=0}^q$ by setting

$$\varepsilon_0 = \varepsilon \text{ and } \varepsilon_{p+1} = \theta_2(\mathcal{F}_p, \varepsilon_p) \tag{28}$$

for all $p = 0, ..., q-1$. Finally we set

$$T\big((\mathcal{F}_p)_{p=0}^q, \varepsilon\big) = \Big\lceil \frac{2}{\varepsilon_q}\cdot B(\mathcal{F}_q, \varepsilon_q/4)\Big\rceil. \tag{29}$$

We then extend T on $\mathcal{R}^{<\mathbb{N}} \times \mathbb{R}_+$ arbitrarily. Let us observe, for later use, that if q is positive then

$$T\big((\mathcal{F}_p)_{p=0}^q, \varepsilon\big) = T\big((\mathcal{F}_p)_{p=1}^q, \theta_2(\mathcal{F}_0, \varepsilon)\big). \tag{30}$$

Although the following notation is quite standard in the literature, we include it below for clarity.

Notation 1. Let $q_0 < q_1 < q_2$ be non-negative integers and $(n_q)_q$ be a sequence of positive integers. Also let $x \in \prod_{p=q_0}^{q_1-1}[n_p]$ and $y \in \prod_{p=q_1}^{q_2-1}[n_p]$. By $x^\frown y$ we denote the concatenation of the sequences x, y, i.e. the sequence $z \in \prod_{p=q_0}^{q_2-1}[n_p]$ satisfying

$z(p) = x(p)$ for all $p = q_0, \dots, q_1 - 1$ and $z(p) = y(p)$ for all $p = q_1, \dots, q_2 - 1$. Moreover, for $A \subseteq \prod_{p=q_0}^{q_1-1}[n_p]$ and $B \subseteq \prod_{p=q_1}^{q_2-1}[n_p]$ we set

$$x^\frown B = \{x^\frown y : y \in B\} \tag{31}$$

and

$$A^\frown B = \bigcup_{x \in A} x^\frown B. \tag{32}$$

The main property of the map T that we are interested in is described by the following lemma. Similar results to this one have already been considered (see [E, ES, GRS]).

Lemma 10. *Let $0 < \varepsilon \leqslant 1$ and q be a non-negative integer. Also set $\mathcal{F}_0, ..., \mathcal{F}_q$ be uniformly regular families and $n_0, ..., n_q$ be integers such that $n_p \geqslant T\big((\mathcal{F}_s)_{s=0}^p, \varepsilon\big)$ for all $p = 0, ..., q$. Finally, let D be a subset of $\prod_{p=0}^q [n_p]$ of density at least ε. Then there exists a sequence $(I_p)_{p=0}^q$ such that*

(i) *I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for all $p = 0, ..., q$ and*
(ii) *$\prod_{p=0}^q I_p$ is subset of D.*

Proof. We proceed by induction on q. First let us observe that for $q = 0$ we have that $T\big((\mathcal{F}_0), \varepsilon\big) \geqslant B(\mathcal{F}_0, \varepsilon)$ and therefore the result follows immediately by the definition the number $B(\mathcal{F}_0, \varepsilon)$.

Assume that the statement holds for some q. Fix a real ε with $0 < \varepsilon \leqslant 1$, uniformly density regular families $\mathcal{F}_0, ..., \mathcal{F}_{q+1}$ and integers $n_0, ..., n_{q+1}$ satisfying $n_p \geqslant T\big((\mathcal{F}_s)_{s=0}^p, \varepsilon\big)$ for all $p = 0, ..., q+1$. Finally, let D be a subset of $\prod_{p=0}^{q+1}[n_p]$ of density at least ε. We set $\Omega = \prod_{p=1}^{q+1}[n_p]$. Observe that $\prod_{p=0}^{q+1}[n_p] = [n_0] \times \Omega$ and that the probability measures $\mathrm{dens}_{\prod_{p=0}^{q+1}[n_p]}$ and $\mathrm{dens}_{n_0} \otimes \mathrm{dens}_\Omega$ are equal. Thus $(\mathrm{dens}_{n_0} \otimes \mathrm{dens}_\Omega)(D) \geqslant \varepsilon$. Since $n_0 \geqslant T\big((\mathcal{F}_0), \varepsilon\big) \overset{(29)}{=} (2/\varepsilon) \cdot B(\mathcal{F}_0, \varepsilon/4)$, by Lemma 7, there exists an element I_0 of $\mathcal{F}_0$ such that setting $\widetilde{D} = \{x \in \Omega : (i, x) \in D \text{ for all } i \in I_0\}$, we have that

$$\mu(\widetilde{D}) \geqslant \theta_2(\mathcal{F}_0, \varepsilon). \tag{33}$$

By the definition of $\widetilde{D}$, it is immediate that

$$I_0^\frown \widetilde{D} \subseteq D. \tag{34}$$

Also notice that for every $p = 1, ..., q+1$,

$$n_p \geqslant T\big((\mathcal{F}_s)_{s=0}^p, \varepsilon\big) \overset{(30)}{=} T\big((\mathcal{F}_s)_{s=1}^p, \theta_2(\mathcal{F}_0, \varepsilon)\big). \tag{35}$$

By (33), (35) and the inductive assumption we have that there exists a sequence $(I_p)_{p=1}^{q+1}$ such that

(a) I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for all $p = 1, ..., q+1$ and
(b) $\prod_{p=1}^{q+1} I_p$ is subset of $\widetilde{D}$.

By (b) and (34), we have that $\prod_{p=0}^{q+1} I_p \subset I_0^\frown \widetilde{D} \subseteq D$ and the proof is complete. $\square$

Definition 11. *Let $0<\varepsilon\leqslant 1$ and r be a non-negative integer. Also let L be a van der Waerden set and $(n_q)_q$ be a sequence of positive integers. We will say that a sequence $(D_\ell)_{\ell\in L}$ is $(r,\varepsilon,(n_q)_q)$-dense if for every $\ell\in L$ with $\ell>r$ we have that D_ℓ is a subset of $\prod_{p=r}^{\ell-1}[n_p]$ of density at least ε.*

For every $0<\varepsilon\leqslant 1$ and every positive integer k we define

$$\theta_3(k,\varepsilon)=\frac{1}{2}\cdot\Big(\frac{\varepsilon}{2}\Big)^{\big\lfloor\frac{2}{\varepsilon\cdot\theta_1(k,\varepsilon)}\big\rfloor}\cdot\theta_1(k,\varepsilon). \tag{36}$$

Finally, for every non-negative integer n and every sequence x of length at least n (finite or infinite), by $R_n(x)$ we denote the initial segment of x of length n.

Lemma 12. *Let $0<\varepsilon\leqslant 1$, r be a non-negative integer and k be a positive integer. Also let $(\mathcal{F}_q)_q$ be a sequence of uniformly density regular families and $(n_q)_q$ be a sequence of positive integers such that $n_q\geqslant T\big((\mathcal{F}_p)_{p=r}^q,\theta_3(k,\varepsilon)\big)$, for all $q\geqslant r$. Finally, let L be a van der Waerden set and $(D_\ell)_{\ell\in L}$ be $(r,\varepsilon,(n_q)_q)$-dense. Then there exist an arithmetic progression P of length k inside L, a van der Waerden subset L' of L, a finite sequence $(I_p)_{p=r}^{r'-1}$ and an $(r',\varepsilon',(n_q)_q)$-dense sequence $(\widetilde{D}_\ell)_{\ell\in L'}$, where $r'=\max P$ and $\varepsilon'=\varepsilon\cdot 2^{-(\prod_{p=r}^{r'-1}n_p+2)}$, satisfying the following.*

(i) *For every $p=r,...,r'-1$, the set I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$.*
(ii) *$r<\min P$.*
(iii) *For every $q\in P$, the set $\prod_{p=r}^{q-1}I_p$ is a subset of D_q.*
(iv) *For every $\ell\in L'$, the set $(\prod_{p=r}^{r'-1}I_p)^\frown\widetilde{D}_\ell$ is a subset of D_ℓ.*

Proof. Passing to a final segment of L, if it is necessary, we may assume that $r<\min L$. Let $\Omega=\prod_{p=r}^{\infty}[n_p]$ and μ be the Lebesgue (probability) measure on Ω. Also let for every $\ell\in L$, $A_\ell=\{x\in\Omega: R_{\ell-r}(x)\in D_\ell\}$. By Lemma 9 applied for "$\eta=\varepsilon$", there exist an arithmetic progression P of length k contained in L, a van der Waerden subset L'' of L and a subset $\widehat{B}$ of $\cap_{i\in P}A_i$ such that

$$\mu(\widehat{B})\geqslant\Big(\frac{\varepsilon}{2}\Big)^{\big\lfloor\frac{2}{\varepsilon\cdot\theta_1(k,\varepsilon)}\big\rfloor-1}\theta_1(k,\varepsilon), \tag{37}$$

for every $\ell\in L''$ we have $\mu_{\widehat{B}}(A_\ell)\geqslant\varepsilon/2$ and the set B belongs to the algebra generated by the family $\{A_\ell:\ell\in L \text{ and } \ell\leqslant\max P\}$. Thus, setting $r'=\max P$, there exists a subset B of $\prod_{p=r}^{r'-1}[n_p]$ such that

$$\mathrm{dens}_{\prod_{p=r}^{r'-1}[n_p]}(B)=\mu(\widehat{B})\overset{(37)}{\geqslant}\Big(\frac{\varepsilon}{2}\Big)^{\big\lfloor\frac{2}{\varepsilon\cdot\theta_1(k,\varepsilon)}\big\rfloor-1}\theta_1(k,\varepsilon), \tag{38}$$

for every $\ell\in L''$, setting $B_\ell=\{x\in\prod_{p=r}^{\ell-1}[n_p]:R_{r'-r}(x)\in B\}$, we have

$$\mathrm{dens}_{B_\ell}(D_\ell)\geqslant\varepsilon/2 \tag{39}$$

and

$$R_{q-r}(x)\in D_q \tag{40}$$

for all $q \in P$ and $x \in B$. Passing to a final subset L'', if it is necessary, we may assume that $\min L'' > r'$.

For every ℓ in L'' we have the following. For each element y in $\prod_{p=r'}^{\ell-1}[n_p]$ we define $\Gamma_y = \{z \in B : z^\frown y \in D_\ell\}$. By (39) and Fubini's Theorem, we have that the set $D'_\ell = \{y \in \prod_{p=r'}^{\ell-1}[n_p] : \mathrm{dens}_B(\Gamma_y) \geqslant \varepsilon/4\}$ is of density at least $\varepsilon/4$ inside $\prod_{p=r'}^{\ell-1}[n_p]$. Since Γ_y is subset of B and therefore subset of $\prod_{p=r}^{r'-1}[n_p]$, we have that there exist a subset Γ_ℓ of B and a subset $\widetilde{D}_\ell$ of D'_ℓ of density at least $(\varepsilon/4)\cdot 2^{-\prod_{p=r}^{r'-1} n_p}$ inside $\prod_{p=r}^{r'-1}[n_p]$ such that $\Gamma_y = \Gamma_\ell$ for all y in $\widetilde{D}_\ell$. Let as observe that by the choice of Γ_ℓ and D'_ℓ we have that

$$\Gamma_\ell^\frown \widetilde{D}_\ell \subseteq D_\ell \text{ and } \mathrm{dens}_B(\Gamma_\ell) \geqslant \varepsilon/4. \tag{41}$$

By Fact 5 there exist a subset Γ of B and a van der Waerden subset L' of L'', such that $\Gamma_\ell = \Gamma$ for all $\ell \in L'$. Clearly, $(\widetilde{D}_\ell)_{\ell\in L'}$ is $(r', \varepsilon', (n_q)_q)$-dense, where ε' is defined in the statement of the lemma. By (36), (38) and (41) we have

$$\mathrm{dens}_{\prod_{p=r}^{r'-1}[n_p]}(\Gamma) = \mathrm{dens}_B(\Gamma)\cdot \mathrm{dens}_{\prod_{p=r}^{r'-1}[n_p]}(B) \geqslant \theta_3(k,\varepsilon). \tag{42}$$

Moreover, by (41), we have that

$$\Gamma^\frown \widetilde{D}_\ell \subseteq D_\ell \tag{43}$$

for all $\ell \in L'$. Since for every $q = r, ..., r'-1$ we have that $n_q \geqslant T\big((\mathcal{F}_p)_{p=r}^q, \theta_3(k,\varepsilon)\big)$, by (42) and Lemma 10 there exists a sequence $(I_p)_{p=r}^{r'-1}$ such that

(a) I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for all $p = r, ..., r'-1$ and
(b) $\prod_{p=r}^{r'-1} I_p$ is subset of Γ.

Since $\prod_{p=r}^{r'-1} I_p \subseteq \Gamma \subseteq B$, by (40) we have that $\prod_{p=r}^{q-1} I_p$ is subset of D_q for all $q \in P$. By (b) and (43) we have that $\prod_{p=r}^{r'-1} I_p^\frown \widetilde{D}_\ell$ is subset of D_ℓ for all $\ell \in L'$ and the proof is complete. □

4. Definition of the map V_δ and the proof of Theorem 2

Let us recall that by $\mathcal{R}$ we denote the set of all uniformly density regular families. For the sequel, let us adopt for following convection. For a sequence of positive integers $(n_q)_q$ and r a non-negative integer, we consider $(n_p)_{p=r}^{r-1}$ to be the empty sequence and $\prod_{p=r}^{r-1} n_p$ to be equal to zero. Fix some real δ with $0 < \delta \leq 1$. We define the map $V_\delta : \mathcal{R}^{<\mathbb{N}} \times \mathbb{N}_+^{<\mathbb{N}} \to \mathbb{N}$ as follows. For every non-negative integer q, every finite sequence $(n_p)_{p=0}^{q-1}$ of positive integers and every finite sequence $(\mathcal{F}_p)_{p=0}^q$ of uniformly density regular families we set

$$V_\delta\big((\mathcal{F}_p)_{p=0}^q, (n_p)_{p=0}^{q-1}\big) = \max_{0\leqslant r\leqslant q} T\Big((\mathcal{F}_p)_{p=r}^q, \theta_3\big(r+1, \delta\cdot 2^{-(\prod_{p=0}^{r-1} n_p + 2r)}\big)\Big) \tag{44}$$

Proof of Theorem 2. Let $(n_q)_q$ be a sequence of positive integers, $(\mathcal{F}_q)_q$ be a sequence of uniformly regular families, L be a var der Waerden set and $(D_\ell)_{\ell\in L}$ be

$(0,\delta,(n_q)_q)$-dense such that

$$n_q \geqslant V_\delta\big((\mathcal{F}_p)_{p=0}^q,(n_p)_{p=0}^{q-1}\big) \tag{45}$$

for every non-negative integer q. We set $L_0=L$, $r_0=0$ and $(D_\ell^0)_{\ell\in L_0}=(D_\ell)_{\ell\in L}$. We inductively construct a sequence of arithmetic progressions $(P_n)_{n=1}^\infty$ contained in L, a decreasing sequence of van der Waerden sets $(L_n)_n$, a sequence $\big((I_q)_{q=r_{n-1}}^{r_n-1}\big)_{n=1}^\infty$ where $r_n=\max P_n$ for all positive integers n and a sequence $\big((D_\ell^n)_{\ell\in L_n}\big)_n$ such that for every non-negative integer n we have the following:

(i) $r_n<\min P_{n+1}$.
(ii) $(D_\ell^n)_{\ell\in L_n}$ is $(r_n,\delta\cdot 2^{-(\prod_{p=0}^{r_n-1}n_p+2r_n)},(n_q)_q)$-dense.
(iii) If n is positive, then I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for every $p=r_{n-1},...,r_n-1$.
(iv) if n is positive, then P_n is an arithmetic progression of length $r_{n-1}+1$ contained in L_{n-1} such that $\prod_{p=0}^{q-1}I_q\subseteq D_q$ for every $q\in P_n$.
(v) $(\prod_{p=0}^{r_n-1}I_p)^\frown D_\ell^n\subseteq D_\ell$ for all $\ell\in L_n$, under the convection $\prod_{p=0}^{-1}I_p=\{\emptyset\}$.

Notice first that for $n=0$ the properties (i)-(v) are satisfied. Assume that for some non-negative integer n we have constructed $(L_m)_{m=0}^n$, $\big((D_\ell^m)_{\ell\in L_m}\big)_{m=0}^n$ and if $n\geqslant 1$ we have constructed $(P_m)_{m=1}^n$ and $\big((I_q)_{q=r_{m-1}}^{r_m-1}\big)_{m=1}^n$ satisfying (i)-(v). Then for every integer q with $q\geqslant r_n$ we have that

$$\begin{aligned} n_q &\overset{(45)}{\geqslant} V_\delta\big((\mathcal{F}_p)_{p=0}^q,(n_p)_{p=0}^{q-1}\big)\\ &\overset{(44)}{\geqslant} T\big((\mathcal{F}_p)_{p=r_n}^q,\theta_3(r_n+1,\delta\cdot 2^{-(\prod_{p=0}^{r_n-1}n_p+2r_n)})\big). \end{aligned} \tag{46}$$

By (46) and the inductive assumption (ii), we have that the assumptions of Lemma 12 for "$\varepsilon=\delta\cdot 2^{-(\prod_{p=0}^{r_n-1}n_p+2r_n)}$", "$k=r_n+1$", "$r=r_n$", "$L=L_n$" and "$(D_\ell)_{\ell\in L}=(D_\ell^n)_{\ell\in L_n}$" are satisfied. Hence there exist an arithmetic progression P_{n+1} of length r_n+1 inside L_n, a van der Waerden subset L_{n+1} of L_n, a finite sequence $(I_p)_{p=r_n}^{r_{n+1}-1}$ and an $(r_{n+1},\varepsilon',(n_q)_q)$-dense sequence $(D_\ell^{n+1})_{\ell\in L_{n+1}}$, where $r_{n+1}=\max P_{n+1}$ and

$$\varepsilon'=\delta\cdot 2^{-(\prod_{p=0}^{r_n-1}n_p+2r_n)}\cdot 2^{-(\prod_{p=r_n}^{r_{n+1}-1}n_p+2)}\geqslant \delta\cdot 2^{-(\prod_{p=0}^{r_{n+1}-1}n_p+2r_{n+1})}, \tag{47}$$

satisfying the following.

(a) For all $p=r_n,...,r_{n+1}-1$, the set I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$.
(b) $r_n<\min P_{n+1}$.
(c) For every $q\in P_{n+1}$, the set $\prod_{p=r_n}^{q-1}I_p$ is a subset of D_q^n.
(d) For every $\ell\in L_{n+1}$, the set $(\prod_{p=r_n}^{r_{n+1}-1}I_p)^\frown D_\ell^{n+1}$ is a subset of D_ℓ^n.

By (c) and the inductive assumption (v) we have for every $q\in P_{n+1}$ that the set $\prod_{p=0}^{q-1}I_p$ is a subset of D_q. By (d) and the inductive assumption (v) we have for every $\ell\in L_{n+1}$ that the set $(\prod_{p=r_n}^{r_{n+1}-1}I_p)^\frown D_\ell^{n+1}$ is a subset of D_ℓ. The proof of the inductive step is complete.

We set $L' = \cup_{n=1}^{\infty} P_n$. Moreover, observe that r_n tends to infinity as n tends to infinity. Thus L' is a van der Waerden set. It is straightforward that L' and $(I_q)_q$ satisfy the conclusion of the Theorem. □

5. Bounds for the map V_δ.

In this section, we are interested in bounds for the map V_δ. For every positive integer m, we denote by $\mathcal{F}_{[m]}$ the family of all subsets of the positive integers with m elements. It is immediate that

$$B(\mathcal{F}_{[m]}, \varepsilon) = \lceil m/\varepsilon \rceil. \tag{48}$$

We set

$$\mathcal{R}_c = \{\mathcal{F}_{[m]} : m \text{ is a positive integer}\}. \tag{49}$$

We will also need the following remark.

Remark 1. Let $\mathcal{R}'$ be a subfamily of $\mathcal{R}$ and $T' : \mathcal{R}'^{<\mathbb{N}} \times \mathbb{R}_+ \to \mathbb{N}$ be a map satisfying the conclusion of Lemma 10 for $\mathcal{F}_0, ..., \mathcal{F}_q$ from $\mathcal{R}'$. Also let $V'_\delta : \mathcal{R}^{<\mathbb{N}} \times \mathbb{N}_+^{<\mathbb{N}} \to \mathbb{N}$ be the map defined as in (44) using T' instead of T. Then one can check that V'_δ satisfies the conclusion of Theorem 2 under the restriction that each $\mathcal{F}_q$ is chosen from $\mathcal{R}'$.

We define $T' : \mathcal{R}_c^{<\mathbb{N}} \times \mathbb{R}_+ \to \mathbb{N}$, setting $T'\big((\mathcal{F}_{[m_p]})_{p=0}^q, \varepsilon\big) = T_\varepsilon\big((\mathcal{F}_{[m_p]})_{p=0}^q\big)$ for every choice of non-negative integer q, positive integers $m_0, ..., m_q$ and real ε with $0 < \varepsilon \leqslant 1$, where T_ε is defined in equation (3) from [TT]. We also define $V'_\delta : \mathcal{R}^{<\mathbb{N}} \times \mathbb{N}_+^{<\mathbb{N}} \to \mathbb{N}$ as in (44) using T' instead of T. Lemma 3 from [TT] yields that T' satisfies the conclusion of Lemma 10. Hence by Remark 1 we have that V'_δ satisfies the conclusion of Theorem 2. For the sequel we fix a sequence $(m_q)_q$ of integers and a real δ satisfying the following.

(i) $0 < \delta \leqslant 1$ and
(ii) $m_q \geqslant 2$ for every non-negative integer q.

We define a maps $f_c : \mathbb{N} \to \mathbb{N}$ inductively as follows. We set

$$f_c(0) = V'_\delta\big((\mathcal{F}_{[m_0]}), \emptyset\big) \text{ and } f_c(q+1) = V'_\delta\big((\mathcal{F}_{[m_p]})_{p=0}^{q+1}, (f_c(p))_{p=0}^q\big) \tag{50}$$

for all $q = 0, 1, ...$ In particular, we are interested in the rate of growth of the map f_c. We will need the following inequalities. By Lemma 14 of [TT] we have that

$$T'\big((\mathcal{F}_{[m_p]})_{p=0}^q, \varepsilon\big) \leqslant A_2\Big(5 \log_2(1/\varepsilon) \prod_{p=0}^q m_p\Big) \tag{51}$$

for every non-negative integer q and every $0 < \varepsilon \leqslant \delta/2$, where $A_2(x) = 2^x$ for every real x. Moreover, by Theorem 18.2 of [G] we have that

$$\mathrm{S}(k, \varepsilon) \leqslant A_2^{(3)}(\log_2(1/\varepsilon) A_2^{(2)}(k+9)), \tag{52}$$

for all positive integers k and all reals ε with $0 < \varepsilon \leqslant 1$.

Proposition 13. *We have that* $f_c(q) \leqslant A_2^{(1+6q)}\big(5\big((2/\delta^2)+1\big)\log_2(2/\delta)m_0+\sum_{p=1}^q m_p\big)$, *for every non-negative integer* q.

Proof. By (2) and (52), for every positive integer k and every $0<\varepsilon\leqslant 1$ we have

$$\begin{aligned}\theta_1(k,\varepsilon) &\geqslant \varepsilon\cdot[A_2^{(3)}(\log_2(2/\varepsilon)A_2^{(2)}(k+9))]^{-2}\\ &\geqslant A_2\big(\log_2(1/\varepsilon)\big)^{-1}\cdot A_2^{(2)}\big(1+A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)^{-1}\\ &\geqslant A_2^{(2)}\big(1+A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)^{-1}\end{aligned} \tag{53}$$

and therefore invoking (36) we have that

$$\begin{aligned}\theta_3(k,\varepsilon) \geqslant& A_2\big(1+(2/\varepsilon)\log_2(2/\varepsilon)A_2^{(2)}\big(1+A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)\big)^{-1}\\ &\cdot A_2^{(2)}\big(1+A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)^{-1}\\ \geqslant& A_2^{(3)}\big(3+A_2\big(\log_2(2/\varepsilon)A_2^{(2)}(k+9)\big)\big)^{-1}.\end{aligned} \tag{54}$$

By (44) and (50), for every positive integer q, we have that

$$\begin{aligned}f_c(q) &\leqslant T'\big((\mathcal{F}_{[m_p]})_{p=0}^q,\theta_3(q+1,\delta\cdot 2^{-(\prod_{p=0}^{q-1}f_c(p)+2q)})\big)\\ &\overset{(51),(54)}{\leqslant} A_2\big(5A_2^{(2)}\big(3+A_2\big(\log_2(2/\varepsilon)A_2^{(2)}(k+9)\big)\big)\prod_{p=0}^q m_p\big)\\ &\leqslant A_2^{(6)}\big(\log_2(2/\delta)+\prod_{p=0}^{q-1}f_c(p)+2q+m_q\big).\end{aligned} \tag{55}$$

By (3) and (36), we have that $\theta_3(1,\varepsilon)\geqslant(\varepsilon/2)^{(2/\varepsilon^2)+1}$, for all $0<\varepsilon\leqslant 1$. Thus by (44), (50) and (51), we have that

$$f_c(0)=T'\big((\mathcal{F}_{[m_0]}),\theta_3(1,\delta)\big)\leqslant A_2\big(5\log_2(2/\delta)(2/\delta^2+1)m_0\big). \tag{56}$$

By inequalities (55), (56) and using induction on q, the result follows. □

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